

SHARPLY 2-TRANSITIVE GROUPS IN CHARACTERISTIC 0

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ABSTRACT. We construct sharply 2-transitive groups of characteristic 0 without non-trivial abelian normal subgroups. These groups act sharply 2-transitively by conjugation on their involutions. This answers a long-standing open question.

1. INTRODUCTION

The finite sharply 2-transitive groups were classified by Zassenhaus [Z] in the 1930's. They were shown to all contain abelian normal subgroups and thus arise from nearfields in a way similar to $AGL(1, K)$ from a field K . It remained an open question whether arbitrary sharply 2-transitive groups necessarily contain an abelian normal subgroup and the first counterexamples were given by [RST]. In the examples constructed there, involutions have no fixed points and are said to have 'characteristic 2', leaving open the existence of non-split sharply 2-transitive groups in other characteristics. The situation when involutions of a sharply 2-transitive group do have fixed points is rather different from the fixed point free setting. In contrast to the result in [RST] which shows that *any* group can be extended to a sharply 2-transitive group in which involutions are fixed point free, it here turns out that a group has to satisfy a number of necessary conditions in order to be embeddable into a sharply 2-transitive group in which involutions have fixed points.

Suppose that G acts sharply 2-transitively on a set X . It is easy to see that the set of involutions J of a sharply 2-transitive group G forms a conjugacy class. If involutions have fixed points, then it is also not hard to see that there is a G -equivariant bijection between J and X taking each involution ι to its unique fixed point $\text{fix}(\iota)$.

Thus, if involutions have fixed points, then G acts sharply 2-transitively on J , implying that all dihedral subgroups of G are isomorphic, see e.g. [RST] and [Te2] for more background.

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We here construct sharply 2-transitive groups of characteristic 0. Here characteristic 0 means that involutions have fixed points and the product of distinct involutions has infinite order, as is the case in $AGL(1, K)$ for a field of characteristic 0.

In order to formulate our main result, we introduce the following terminology: given an involution $j \in G$ we say that involutions $s, t \in G$ are *equivalent relative to j* (and write $s \sim_j t$) if there are $n, m \in \mathbb{Z}$ such that $(js)^m = (jt)^n$. Note that any involution $s \in \langle j, t \rangle \setminus \{j\}$ is equivalent to t relative to j : we have $s = (jt)^n j$ or $s = j(jt)^n$ for some $n \in \mathbb{N}$. Hence $js = (jt)^{-n}$ or $js = (jt)^n$, respectively. Since j will be fixed throughout this paper, we mostly omit mentioning j in the equivalence relation. We call an involution s_1 *minimal* if for any involution s_2 equivalent to s_1 (relative to j) we have $\langle j, s_1 \rangle \geq \langle j, s_2 \rangle$.

For any involution $s \in G$, we write $\overline{\langle j, s \rangle}^G = \bigcup_{s' \sim_j s} \langle j, s' \rangle$. Hence, if $s \in G$ is minimal relative to j , then $\overline{\langle j, s \rangle}^G = \langle j, s \rangle$. We omit mentioning G if the ambient group is clear from the context. By a *translation* we will mean a product of two involutions. For a set $A \subset G$ we write $\text{Tr}(A)$ for the set of translations in A . Note that we have $\text{Tr}(\overline{\langle j, s \rangle}) = \{1\} \cup \{js' : s' \text{ equivalent to } s\}$.

Our main result is the following:

Theorem 1.1. *Let G be a group containing involutions j, t and t' with $jt' = t$. Let $A = \text{Cen}_G(j)$ and assume that the following holds:*

- (1) *all involutions in G are conjugate;*
- (2) *any two distinct involutions of G generate infinite dihedral groups;*
- (3) *for any $n > 0$ there is a unique involution $s \in G$ with $(js)^n = jt$.*
- (4) *for any involution s equivalent to t there is some $a \in A$ such that $s = t^a$.*
- (5) *for any involution s , we have $\text{Cen}(js) = \text{Tr}(\overline{\langle j, s \rangle})$.*
- (6) *for any involution $s \notin AtA$ there is a unique minimal involution s' equivalent to s relative to j .*

Then G is contained in a group \mathcal{G} acting sharply 2-transitively and in characteristic 0 on the set of involutions of \mathcal{G} and such that \mathcal{G} contains no abelian normal subgroup.

As an immediate consequence of Theorem 1.1 we have

Theorem 1.2. *There exist sharply 2-transitive groups of characteristic 0 not containing a non-trivial abelian normal subgroup.*

Theorem 1.1 will follow inductively from the following proposition, which states that any Zassenhaus group can be extended to a Zassenhaus group in which the point stabilizers are ‘a bit more transitive’:

Proposition 1.3. *Let G be a group containing involutions j, t and t' with $jt' = t$. Let $A = \text{Cen}_G(j)$ and assume that conditions (1) – (6) of Theorem 1.1 hold.*

Then for any involution $v \in G$ with $v \neq j$ there exist an extension G_1 of G such that for $A_1 = \text{Cen}_{G_1}(j)$ there exists some $f \in A_1$ with $t^f = v$ and conditions (1) – (6) continue to hold with G_1 and A_1 in place of G, A .

The main body of the paper is concerned with the proof of Proposition 1.3. Theorems 1.1 and 1.2 then follow rather quickly from this. Their proofs will be given in Section 6.

It might be worth pointing out that for a group G to be contained in a sharply 2-transitive group of characteristic 0, conditions (2) and (5) of Theorem 1.1 are necessary, while assumption (3) is necessary in the following weaker form:

(3') for any $n > 0$ there is at most one involution $s \in G$ with $(js)^n = jt$.

Assumptions (1), (4) and (6) will necessarily hold in any sharply 2-transitive group of characteristic 0. They are added here as prerequisites so as to facilitate the proof of Proposition 1.3, which forms the induction step for the proof of Theorem 1.1.

We would like to point out that in the case where involutions have fixed points, the approach using partial actions as in [TZ, Te1] leads to exactly the same situation to deal with as in the approach used here: since any sharply 2-transitive action of G on a set X in characteristic different from 2 is necessarily (equivalent to) the conjugation action on the set of involutions of G and the involutions generate subgroups, one does not gain any freedom in the construction from using partial actions.

Finally we mention that while the sharply 2-transitive groups of characteristic 2 constructed in [TZ] were shown in [Te1] to be point stabilizers of sharply 3-transitive groups (of the same characteristic) we do not know whether such an extension is possible for the groups constructed here.

2. SOME PRELIMINARIES ON INVOLUTIONS

In this section we collect some facts about involutions and the equivalence relation relative to the fixed involution $j \in A$ defined above. Throughout this section we assume that $G, j, A = \text{Cen}_G(j), t$ and t' are as in the statement of Theorem 1.1 and satisfy the assumptions (1) – (6) given there.

Recall that a proper subgroup B of a group H is *malnormal* in H if $B \cap g^{-1}Bg = 1$, for all $g \in H \setminus B$.

Lemma 2.1. (i) The centralizer $A = \text{Cen}(j)$ is malnormal in G .

(ii) If $g^n \in A$, then $g \in A$ or $g^n = 1$.

(iii) The intersection of A with any dihedral subgroup of G is contained in $\langle j \rangle$. In particular, j is the unique involution contained in A .

(iv) If $r, s \in G \setminus A$ are involutions with $s = arb$ for some $a, b \in A$, then $a = b^{-1}$.

Proof. (i) Suppose $g \in G \setminus A$ is such that $A \cap A^g \neq 1$. Since $A^g = \text{Cen}_G(j^g)$, any $h \in A \cap A^g$ centralizes both j and $j^g \neq j$. Hence h centralizes the translation jj^g and therefore is contained in $\text{Tr}\langle j, j^g \rangle$. By Property (5) no nontrivial translation in $\text{Tr}\langle j, j^g \rangle$ centralizes j , a contradiction.

(ii) If $g^n \in A$, then $g^n \in A \cap A^g$, and thus by malnormality of A we have $g^n = 1$ or $g \in A$.

(iii) Since $A = \text{Cen}_G(j)$ is malnormal and all involutions are conjugate in G , it follows that j is the unique involution in A . Now if s, r are distinct involutions in G such that $sr \in A$, we may assume that $s \notin A$. Since s inverts rs , we contradict the malnormality of A in G .

(iv) We have $s = ara^{-1}ab$. Hence $ab = ara^{-1}s$ is a product of two involutions. By (iii), $ab = 1$. \square

Lemma 2.2. *If $1 \neq s_1s_2 \in \langle j, s_3 \rangle$ for involutions $s_1, s_2, s_3 \neq j$, then s_1 and s_2 are equivalent to s_3 relative to j .*

Proof. We may suppose $s_1s_2 = js_3$. Conjugation by j inverts both sides and hence yields $s_1^j s_2^j = s_2s_1 = s_3j$. Hence $s_2s_1^j s_2^j = s_1 = s_2^j s_1^j s_2$.

Conjugating the left and the right part of the previous equation by s_2 we see that

$$s_1^j s_2^j s_2 = s_2 s_2^j s_1^j.$$

Multiplying both sides of the equation from the right by s_1^j we have

$$s_1^j s_2^j s_2 s_1^j = s_2 s_2^j.$$

Thus s_1^j inverts $s_2 s_2^j$. Conjugating by j we see that also s_1 inverts $s_2 s_2^j$. Hence $s_1^j s_1 = js_1 js_1 = (js_1)^2$ centralizes $s_2^j s_2 = (js_2)^2$. By assumption, s_1 and s_2 are equivalent relative to j . Since we also have $s_3 s_2 = js_1$, we see similarly that s_2 and s_3 are equivalent relative to j , proving the claim. \square

Lemma 2.3. *For any distinct involutions $s_1, s_2 \in G$ with $s_1s_2 \in \langle j, t \rangle$ there is some $g \in G$ with $s_1^g = j, s_2^g = t$.*

Proof. We may assume that $s_2 \neq j$. Then s_2 is equivalent to t by Lemma 2.2. By assumption (4) there is some $a \in A$ with $s_2^a = t$. Note that $A^{t'}$ is the centralizer of t . Since the involutions in $\langle j, t \rangle \setminus \{t\}$ are equivalent to j relative to t , they are conjugate under $A^{t'}$ by property (4) applied to $A^{t'} = \text{Cen}(t)$. Hence there is some $g \in A^{t'}$ with $(s_1^a)^g = j$ and $ag \in G$ is as required. \square

We will use the following easy facts:

Lemma 2.4. *Suppose $v \in G$ is a minimal involution and s is equivalent to v relative to j . If $(js)^g \in \langle j, v \rangle$ for some $g \in G$, then $g \in \langle j, v \rangle$.*

Proof. Since ju centralizes js , we see that $(ju)^g$ centralizes $(js)^g \in \langle ju \rangle$. Hence $(ju)^g \in \langle ju \rangle$ showing that g normalizes $\langle ju \rangle$ and hence either centralizes or inverts ju . Now $j \in \langle j, v \rangle$ inverts ju and $\text{Cen}(ju) = \langle ju \rangle$, proving the claim. \square

Lemma 2.5. *For any involution $s \in G$ and $n > 0$, there is at most one involution s_1 such that $(js_1)^n = js$. In particular, $(js_1)^n = (js_2)^n$ implies $s_1 = s_2$.*

Proof. If $s \notin AtA$, this holds in $\langle j, s' \rangle$ for s' the minimal involution equivalent to s . If $s \in AtA$, then s is conjugate to t by Lemma 2.1 and the claim follows from Property (3). \square

3. BACKGROUND ON HNN-EXTENSIONS

The construction of G_1 in Proposition 1.3 will be given in Section 4 as an HNN-extension of G . In preparation for the proof we here collect some general facts about HNN-extensions. We start with Britton's lemma, which we state for an arbitrary group G with isomorphic subgroups $D_t, D_v \leq G$ and a fixed isomorphism $f : D_t \rightarrow D_v$.

Suppose $D_t, D_v \leq G$ and $G_1 = \langle G, f \mid D_t^f = D_v \rangle$ is an HNN extension of G . Then any element of G_1 has the form

$$g = g_0 f^{\delta_1} g_1 \cdots g_{m-1} f^{\delta_m} g_m,$$

where $g_i \in G$, $i = 0, \dots, m$, $\delta_i = \pm 1$, $i = 1, \dots, m$. We say that the expression for g is *reduced* if the equality $\delta_i = -\delta_{i-1} = 1$ implies $g_i \notin D_t$, and $\delta_i = -\delta_{i-1} = -1$ implies $g_i \notin D_v$. Thus, an expression for g is reduced if there are *no f -cancellations* in g , and in that case we call m the *length* of g .

Britton's lemma can be phrased in the following way:

Theorem 3.1. (Britton's lemma) If $g \in G_1 = \langle G, f \mid D_t^f = D_v \rangle$ has a reduced expression

$$g = g_0 f^{\delta_1} g_1 \cdots g_{m-1} f^{\delta_m} g_m$$

then *every* reduced expression for g is of the form

$$g = g_0 w_1 f^{\delta_1} z_1^{-1} g_1 w_2 \cdots g_{m-1} w_m f^{\delta_m} z_m^{-1} g_m$$

where

$$w_i \in D_t \text{ if } \delta_i = 1 \text{ and } z_i = w_i^f \in D_v$$

and

$$w_i \in D_v \text{ if } \delta_i = -1 \text{ and } z_i = w_i^{f^{-1}} \in D_t.$$

In particular, the length of an element in G_1 is well-defined.

Remark 3.2. Suppose $G_1 = \langle G, f \mid D_t^f = D_v \rangle$ is an HNN extension of G and $g, h \in G_1$ have reduced expressions

$$g = g_0 f^{\delta_1} g_1 \cdots g_{m-1} f^{\delta_m} g_m$$

and

$$h = h_0 f^{\eta_1} h_1 \cdots h_{k-1} f^{\eta_k} h_k.$$

If gh has length $m + k - n$, then it follows from Britton's lemma that n is even and that we can rewrite h as

$$h = g_m^{-1} f^{-\delta_m} g_{m-1}^{-1} \cdots g_{m-n/2}^{-1} f^{-\delta_{m-n/2}} h'_{n/2+1} \cdots h_{k-1} f^{\eta_k} h_k$$

where $h'_{n/2+1} = wh_{n/2+1}$ for some $w \in D_t \cup D_v$.

As a special case we note the following for future reference:

Remark 3.3. Any involution $s \in G_1$ has a reduced expression of the form

$$s = g_1 f^{\epsilon_1} g_2 f^{\epsilon_2} \dots g_m f^{\epsilon_m} s_1 f^{-\epsilon_m} g_m^{-1} \dots g_2^{-1} f^{-\epsilon_1} g_1^{-1}$$

for some involution $s_1 \in G$.

Definition 3.4. We call an element $g \in G_1 = \langle G, f \mid D_t^f = D_v \rangle$ *cyclically reduced* if any reduced expression for g of the form

$$g = at^{\epsilon_1} g_2 f^{\epsilon_2} \dots f^{\epsilon_k} a^{-1}$$

implies $a = 1$ and $\epsilon_1 = \epsilon_k$.

As a convenient abbreviation we will write

$$D_1 = D_t \text{ and } D_{-1} = D_v$$

so that for $\epsilon \in \{-1, 1\}$ we have

$$D_\epsilon f^\epsilon = f^\epsilon D_{-\epsilon}.$$

Remark 3.5. (i) Clearly, any $g \in G_1$ is conjugate to a cyclically reduced one. In particular, any $g \in G$ is cyclically reduced.

(ii) If

$$g = g_1 f^{\epsilon_1} g_2 f^{\epsilon_2} \dots f^{\epsilon_k} g_{k+1}$$

is cyclically reduced, then so is

$$g^{g_{k+1}^{-1}} = g_{k+1} g_1 f^{\epsilon_1} g_2 f^{\epsilon_2} \dots f^{\epsilon_k}.$$

Otherwise we have $h = g_{k+1} g_1 \in D_{\epsilon_1}$ and $\epsilon_1 = -\epsilon_k$. Thus $g_{k+1} = h g_1^{-1}$ and we have a reduced expression for g of the form

$$g = g_1 f^{\epsilon_1} g_2 f^{\epsilon_2} \dots g_k h f^{-\epsilon_k} f^{\epsilon_k} g_1^{-1},$$

contradicting our assumption that g be cyclically reduced.

(iii) Note that if g is cyclically reduced, then for any $n \in \mathbb{N}$ and any reduced expression for g , we obtain a reduced expression for g^n by concatenating n copies of the reduced expression for g .

Lemma 3.6. Suppose $G_1 = \langle G, f \mid D_t^f = D_v \rangle$ and let $g \in G, h \in G_1$ be such that $g^h \in G$. Then we have $h \in G$ or there is some $b \in G$ with $g^b \in D_t \cup D_v$. In particular, if the centralizer of g is not contained in G , then there is some $b \in G$ with $g^b \in D_t \cup D_v$.

Proof. Write $h \in G_1$ in reduced form,

$$h = h_1 f^{\delta_1} h_2 \dots f^{\delta_m} h_{m+1}.$$

If $h \notin G$ and $g^h \in G$, then f -cancellation must occur at $f^{-\delta_1} h_1^{-1} g h_1 f^{\delta_1}$ implying that $g^{h_1} \in D_t \cup D_v$ with $h_1 \in G$. \square

4. CONSTRUCTION OF G_1 AND A_1 IN PROPOSITION 1.3

With the notation as in Theorem 1.3 note that if there is an element $f \in A$ such that $t^f = v$, we can just take $G_1 = G$ and $A_1 = A$ and there is nothing to prove in Proposition 1.3. Moreover, we may assume that v is a minimal involution relative to j :

Lemma 4.1. *It suffices to consider the case that v is a minimal involution relative to j .*

Proof. Suppose we have constructed (G_1, A_1) as in Proposition 1.3 for a minimal involution v . Then v is conjugate under A_1 to t . Since all involutions equivalent to t are conjugate under $A \leq A_1$, all involutions equivalent to v will be conjugate under A_1 . \square

So let v be a minimal involution relative to j and put

$$D_t = \langle j, t \rangle \text{ and } D_v = \langle j, v \rangle.$$

Note that D_t and D_v are infinite dihedral groups and hence isomorphic under an isomorphism f fixing j and taking t to v .

We may now define the extension G_1 required in Proposition 1.3 as the HNN-extension of G by f taking D_t to D_v , i.e.

$$G_1 = \langle G, f \mid j^f = j, t^f = v \rangle.$$

Proposition 1.3 will be proved once we show that G_1 and $A_1 = \text{Cen}_{G_1}(j)$ satisfy the required properties in Proposition 1.3. Clearly, $f \in A_1$ and $t^f = v$.

5. PROPERTIES (1) – (6) HOLD IN G_1

We have to verify that G_1 and $A_1 = \text{Cen}_{G_1}(j)$ satisfy the conditions of Proposition 1.3:

- (1) all involutions in G_1 are conjugate;
- (2) any two distinct involutions of G_1 generate infinite dihedral groups;
- (3) for any $n > 0$ there is a unique involution $s \in G_1$ with $(js)^n = jt$.
- (4) for any involution $s \in G_1$ equivalent to t there is some $a \in A_1$ such that $s = t^a$.
- (5) for any involution $s \in G_1$, we have $\text{Cen}(js) = \text{Tr}(\overline{\langle j, s \rangle}^{G_1})$.
- (6) for any involution $s \notin A_1 t A_1$ there is a unique minimal involution s' equivalent to s relative to j .

It is well-known that Properties (1) and (2) are preserved under HNN-extensions. For the remaining properties, we collect some easy observations:

Lemma 5.1. *The involution j is the only involution in A_1 .*

Proof. Suppose $s \in A_1 \setminus G$ is an involution. Write s in reduced form as

$$s = g_1 f^{\epsilon_1} g_2 f \dots g_m f^{\epsilon_m} s_1 f^{-\epsilon_m} g_m^{-1} \dots g_2^{-1} f^{-\epsilon_1} g_1^{-1}$$

for some involution $s_1 \in G$. Then

$$sjs = g_1 f^{\epsilon_1} \dots g_m f^{\epsilon_m} s_1 f^{-\epsilon_m} g_m^{-1} \dots f^{-\epsilon_1} g_1^{-1} j g_1 f^{\epsilon_1} \dots g_m f^{\epsilon_m} s_1 f^{-\epsilon_m} g_m^{-1} \dots f^{-\epsilon_1} g_1^{-1} = j$$

Clearly f -cancellation must occur at $f^{-\epsilon_1} g_1^{-1} j g_1 f^{\epsilon_1}$. We conclude inductively that $j_1 = j g_1 f^{\epsilon_1} g_2 f^{\epsilon_2} \dots g_m f^{\epsilon_m} \in D_{-\epsilon_m}$ and $s_1 \in \text{Cen}(j_1)$. By assumption on G we have $s_1 = j_1 \in D_{-\epsilon_m}$, contradicting our assumption that the expression for $s \in G_1 \setminus G$ was reduced. \square

Lemma 5.2. *If $g \in D_t$ is a nontrivial translation, then $g^b \notin D_v$ for all $b \in G$.*

Proof. By Lemma 2.3 it suffices to consider $g = jt$. Suppose $(jt)^b \in D_v$. By replacing b by bj if necessary, we may assume $(jt)^b = (jv)^n$ for some $n > 0$. By properties (3) and (4) there is some $a \in A$ be such that $(jt^a)^n = jt$, so $(jt)^{ab} = jv$ by Lemma 2.5. By Lemma 2.3 there is some $g \in G$ such that

$$j^{b^{-1}a^{-1}g} = j \text{ and } v^{b^{-1}a^{-1}g} = t.$$

Then $h = g^{-1}ab \in \text{Cen}(j) = A$ and $t^h = v$, contradicting the assumption on v . \square

Lemma 5.3. *For any translation $w \in \langle j, t \rangle$ the centralizer of w in G_1 coincides with the centralizer of w in G . In particular we have*

$$\text{Cen}_G(w) = \text{Tr}(\overline{\langle j, t \rangle}^G) = \text{Tr}(\overline{\langle j, t \rangle}^{G_1}) = \text{Cen}_{G_1}(w).$$

Proof. By Lemma 2.3 we may assume $w = jt$. Let $g \in \text{Cen}_{G_1}(jt) \setminus G$ have a reduced expression

$$g = g_1 f^{\epsilon_1} g_2 f^{\epsilon_2} \dots f^{\epsilon_k} g_{k+1}.$$

and suppose towards a contradiction that $k > 0$. Then

$$jt = g_{k+1}^{-1} f^{-\epsilon_k} g_k^{-1} \dots f^{-\epsilon_1} g_1^{-1} j t g_1 f^{\epsilon_1} g_2 f^{\epsilon_2} \dots f^{\epsilon_k} g_{k+1}. \quad (*)$$

By Britton's lemma, cancellation must occur at

$$f^{-\epsilon_1} g_1^{-1} j t g_1 f^{\epsilon_1}$$

implying that $g_1^{-1} j t g_1 \in D_t$ by Lemma 5.2 and hence $\epsilon_1 = 1$.

Thus we have

$$f^{-\epsilon_1} g_1^{-1} j t g_1 f^{\epsilon_1} \in D_v.$$

Again by Lemma 5.2 we must have $k \geq 2$. Since all the f -occurrences in $(*)$ cancel, by Lemma 5.2 and Lemma 2.4 we must have $g_2 \in D_v$ and $\epsilon_2 = -1$, contradicting the assumption that the expression for g was reduced. \square

Corollary 5.4. *If $s \in G_1$ is equivalent to t relative to j , then $s \in G$.*

Proof. Let $s \in G_1$ be such that $(js)^m = (jt)^n$. Then js centralizes jt , so $js \in G$ by Lemma 5.3 and hence $s \in G$. \square

Clearly, Corollary 5.4 implies that Properties (3) and (4) continue to hold in G_1 :

Corollary 5.5. (i) For any $n > 0$ there is a unique involution $s \in G_1$ with $(js)^n = jt$.

(ii) For any involution $s \in G_1$ equivalent to t there is some $a \in A_1$ such that $s = t^a$. \square

We next verify Property (5) by a variant of the *Euclidean algorithm*. Recall that we write

$$D_1 = D_t \text{ and } D_{-1} = D_v$$

so that for $\epsilon \in \{-1, 1\}$ we have

$$D_\epsilon f^\epsilon = f^\epsilon D_{-\epsilon}.$$

We first need the following lemma:

Lemma 5.6. Let $r \in G, s \in G_1$ be involutions. Then there is a cyclically reduced translation $r's'$ conjugate to rs where $r' \in G, s' \in G_1$ are involutions.

Proof. Write $s \in G_1$ in reduced form

$$s = s_1 f^{\epsilon_1} s_2 f^{\epsilon_2} \dots s_m f^{\epsilon_m} s' f^{-\epsilon_m} g_m^{-1} \dots s_2^{-1} f^{-\epsilon_1} s_1^{-1}.$$

If rs is not cyclically reduced, then $w_1 = s_1^{-1} r s_1 \in D_{\epsilon_1}$. Put $z_1 = w_1^{f^{\epsilon_1}}$. Then

$$(rs)^{s_1 f^{\epsilon_1}} = f^{-\epsilon_1} w_1 f^{\epsilon_1} s_2 f^{\epsilon_2} \dots f^{-\epsilon_2} s_2^{-1} = z_1 s_2 f^{\epsilon_2} \dots f^{-\epsilon_2} s_2^{-1}.$$

By iterating this step, the length decreases and we end with a cyclically reduced conjugate of rs in the required form. \square

Now we can show that Property (5) continues to hold in G_1 :

Lemma 5.7. For involutions $r \in G, s \in G_1$, we have $\text{Cen}(rs) = \text{Tr}(\overline{\langle r, s \rangle}^{G_1})$.

Proof. By Lemma 5.6 we may assume that rs is cyclically reduced. If $rs \in G$, the claim holds by Lemmas 3.6 and 5.3 and assumption on G . Hence we may assume $rs \in G_1 \setminus G$.

Write rs as a reduced expression

$$rs = r g_1 f^{\epsilon_1} g_2 f^{\epsilon_2} \dots g_m f^{\epsilon_m} s_1 f^{-\epsilon_m} g_m^{-1} \dots g_2^{-1} f^{-\epsilon_1} g_1^{-1}$$

for involutions $r, s_1 \in G$. Note that by Remark 3.5 we may also assume $g_1 = 1$ and so $r \notin D_{\epsilon_1}$. Consider $h \in \text{Cen}(rs)$.

If $h \in G$, then by Britton's lemma and the equality $rsh = hrs$ we have $h, h^r \in D_{\epsilon_1}$. If h is a translation, then by Lemma 5.3 we have $h \in D_v$, so $\epsilon_1 = -1$. By Lemma 2.4 again $r \in D_v = D_{\epsilon_1}$, contradicting our assumption on rs . Hence $h \in D_{\epsilon_1}$ is an involution centralizing rs . Since

$$rhrsh = s = hsrhr,$$

we see that the translation $h^r h = rhrh$ is inverted by s and hence centralized by $js \in G_1 \setminus G$. By Lemma 5.3 we have $h^r h \in D_v$, so again $\epsilon_1 = -1$. Now

$(hh^r)^r = h^r h$ and so r inverts hh^r . Lemma 2.4 implies $r \in D_v = D_{\epsilon_1}$, contradicting our assumption on r .

Hence $h \in G_1 \setminus G$. Since rs is cyclically reduced, by considering h^{-1} if necessary, we may assume that h has no reduced expression starting with f^{ϵ_1} . The lemma will follow from:

Claim: There is an involution $t_1 \in G_1 \setminus G$ of length $d = \gcd(2m, k)$ such that $h, rs \in \langle rt_1 \rangle$.

Write $h = h_1 f^{\delta_1} h_2 \dots h_k f^{\delta_k} h_{k+1}$. By assumption on h and Remark 3.2 the expression

$$rsh = r f^{\epsilon_1} g_2 f^{\epsilon_2} \dots g_2^{-1} f^{-\epsilon_1} h_1 f^{\delta_1} h_2 \dots f^{\delta_k} h_{k+1}$$

is reduced and since $rsh = hrs$, by Britton's lemma so is

$$hrs = h_1 f^{\delta_1} h_2 \dots f^{\delta_k} h_{k+1} r f^{\epsilon_1} g_2 f^{\epsilon_2} \dots g_2^{-1} f^{-\epsilon_1}.$$

Again by Britton's lemma, we see that $\delta_1 = \epsilon_1 = -\delta_k$. In particular $k \geq 2$ and we may assume $h_1 = r, h_{k+1} = 1$. So h is cyclically reduced as well. Now consider

$$\begin{aligned} rsh &= r f^{\epsilon_1} g_2 f^{\epsilon_2} \dots g_2^{-1} f^{-\epsilon_1} r f^{\epsilon_1} h_2 f^{\delta_2} \dots h_k f^{-\epsilon_1} = \\ hrs &= r f^{\epsilon_1} h_2 f^{\delta_2} \dots h_k f^{-\epsilon_1} r f^{\epsilon_1} g_2 f^{\epsilon_2} \dots g_2^{-1} f^{-\epsilon_1}. \end{aligned}$$

Write $k = n \cdot 2m + l$ with $l < 2m$. We can rewrite h according to Britton's lemma as

$$h = r f^{\epsilon_1} g_2 \dots f^{-\epsilon_1} r f^{\epsilon_1} \dots f^{-\epsilon_1} r f^{\epsilon_1} \dots f^{-\epsilon_1} r f^{\epsilon_1} g_2 f^{\epsilon_2} \dots f^{\epsilon_l} w.$$

where $w \in D_{-\epsilon_l}$. If $l = 0$, the claim follows. Otherwise $h' = (rs)^{-n} h \in \text{Cen}(rs)$ has a reduced expression

$$h' = r f^{\epsilon_1} g_2 f^{\epsilon_2} \dots f^{\epsilon_l} w$$

where $\epsilon_l = -\epsilon_1, \epsilon_{l-1} = -\epsilon_2$ etc.

Since $r \notin D_{\epsilon_1}$, we see that $h' \in \text{Cen}(rs)$ is again cyclically reduced. We can therefore iterate the procedure with h', rs in place of rs, h , rewriting rs in reduced form as a concatenation of copies of the reduced expression for h' followed by an initial segment of h' and r .

This procedure stops with a reduced expression of length $d = \gcd(m, k) \geq 2$ of the form

$$h'' = r f^{\epsilon_1} g_2 \dots g_2^{-1} f^{-\epsilon_1} \in \text{Cen}(rs).$$

So $h'' = rt_1$ for some involution $t_1 \in G_1$ equivalent to s and of length d with and $h, rs \in \langle rt_1 \rangle$. \square

Note the following corollary of the proof:

Corollary 5.8. *If $s \in G_1 \setminus G$ is such that js is cyclically reduced, then there is an involution $t_1 \in G_1 \setminus G$ such that jt_1 is cyclically reduced and $\text{Cen}(js) = \langle jt_1 \rangle$. In particular, t_1 is the unique minimal involution equivalent to s .*

Proof. All the claims follow from the proof of Lemma 5.7: the proof shows that any element in the centralizer of js has length at least 2. Hence an element of minimal length in the centralizer will yield the required minimal involution. \square

Now we can also establish Property (6):

Lemma 5.9. *If $s \in G_1$ is an involution with $s \notin A_1tA_1$, then there is a unique minimal involution s' equivalent to s relative to j .*

Proof. By conjugating we may assume that js is cyclically reduced. If $s \in G$, then the result holds by Lemmas 3.6 and 5.3. If $s \in G_1 \setminus G$, then $s \notin A_1tA_1$ and this is contained in Corollary 5.8. \square

6. THE PROOF OF THEOREMS 1.1 AND 1.2

In this section we show how Theorems 1.1 and 1.2 of the introduction follows from Proposition 1.3.

In order to obtain a new group from G and A in Proposition 1.3, we need that G is not already sharply 2-transitive. Thus to get started, we can use the following preparatory lemma:

Lemma 6.1. *If G_0 is a group containing involutions j, t and t' such that the assumptions of Theorem 1.1 hold, then $G = G_0 * \mathbb{Z}$ also satisfies the assumptions. If G_0 acts sharply 2-transitively on the set of its involutions, then G is not 2-transitive on the set of its involutions.*

Proof. It follows as before that conditions (1), (2), (3) and (4) are preserved. It remains to verify (5) and (6). The proofs are exactly as the corresponding proofs in Section 5. Note that $\text{Cen}_G(j) = \text{Cen}_{G_0}(j)$, and the set of involutions of G_0 is strictly contained in the corresponding set of G . So the last claim is immediate. \square

Proof. (of Theorem 1.1) Let G, A and j, t and t' be as in Theorem 1.1. If G is already sharply 2-transitive, consider $G * \mathbb{Z}$ instead. Set $G_0 := G$, $A_0 := A$. We now use Proposition 1.3 to construct an ascending sequence of groups G_i such that with $A_i = \text{Cen}_{G_i}(j)$ the groups G_i, A_i satisfy properties (1) – (6) of Theorem 1.1 and such that for each $i \geq 0$, and each involution $v_i \in G_i, v_i \neq j$ there exists an element $f_i \in A_{i+1}$ such that $t^{f_i} = v_i$.

Suppose the groups G_k and their subgroups A_k were constructed for $k \leq i$. We then construct G_{i+1} as the union of groups obtained from applying Proposition 1.3 to each minimal involution $v_i \in G_i, v_i \neq j$. Then for any involution $v \in G_i$ there exists an element $f \in A_{i+1} = \text{Cen}_{G_{i+1}}(j)$ with $t^f = v$.

Put $\mathcal{G} = \bigcup_{i=0}^{\infty} G_i$. Then $\text{Cen}_{\mathcal{G}}(j) = \mathcal{A} := \bigcup_{i=0}^{\infty} A_i$ and \mathcal{G} acts sharply 2-transitive on the set of its involutions. It is left to show that \mathcal{G} does not contain a non-trivial abelian normal subgroup. Otherwise by a result of B. H. Neumann [Ne] such a subgroup would be of the form $\text{Tr}(\mathcal{G})$, which is

nonabelian by property (4) whenever \mathcal{G} contains distinct involutions j, s, t with t not normalizing $\langle j, s \rangle$ and these exist in \mathcal{G} by construction. \square

We finish the paper with the proof of Theorem 1.2:

Proof. (of Theorem 1.2) The sharply 2-transitive group $AGL(1, \mathbb{Q}) \cong \mathbb{Q}^+ \rtimes \mathbb{Q}^*$ and $A = \mathbb{Q}^*$ satisfies the assumptions of Theorem 1.1. By Lemma 6.1 we may therefore apply Theorem 1.1 to

$$G = AGL(1, \mathbb{Q}) * \mathbb{Z}$$

and obtain a sharply 2-transitive group without abelian normal subgroup. \square

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